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DIFFUSION AND LAPLACIAN TRANSPORT FOR ABSORBING DOMAINS

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Abstract

We study (stationary) Laplacian transport by the Dirichlet-to-Neumann formalism. Our results concerns a *formal* solution of the *geometrical* inverse problem for localisation and reconstruction of the form of absorbing domains. Here we restrict our analysis to the one- and two-dimension cases. We show that the last case can be studied by the conformal mapping technique. To illustrate it we scrutinize constant boundary conditions and analyse a numeric example.

Key words: Laplacian transport, Dirichlet-to-Neumann operators, Conformal mapping.
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1. Introduction

1. Is is known (see e.g. [LeUl]) that the problem of determining a *conductivity matrix* field $\gamma(p) = [\gamma_{i,j}(p)]_{i,j=1}^{d}$, for p in a bounded open domain $\Omega \subset \mathbb{R}^d$, is related to "measuring" the elliptic *Dirichlet-to-Neumann* map for associated conductivity equation. Notice that solution of this problem has a lot of practical applications in various domains: geophysics, electrochemistry etc. It is also an important diagnostic tool in medicine, e.g. in the *electrical impedance tomography*; the tissue in the human body is an example of highly anisotropic conductor [BaBr].

Under assumption that there is no sources or sinks of current the potential $v(p),\ p \in \Omega$, for a given voltage $f(\omega),\ \omega \in \partial\Omega$, on the (smooth) boundary $\partial\Omega$ of Ω is a solution of the Dirichlet problem:

$$(\mathbf{P1}) \qquad \begin{cases} \operatorname{div}(\gamma \nabla v) = 0 & \text{in } \Omega, \\ v|_{\partial\Omega} = f & \text{on } \partial\Omega. \end{cases}$$

Then the corresponding to (**P1**) Dirichlet-to-Neumann map (operator) $\Lambda_{\gamma,\partial\Omega}$ is (*formally*) defined by [Tay1]

$$\Lambda_{\gamma,\partial\Omega} : f \mapsto \partial v_f/\partial \nu_\gamma := \nu \cdot \gamma\ \nabla v_f\ |_{\partial\Omega}\ . \tag{1.1}$$

Here ν is the unit *outer-normal* vector to the boundary at $\omega \in \partial\Omega$ and the function $v := v_f$ is a solution of the Dirichlet problem (**P1**).

The Dirichlet-to-Neumann operator (1.1) is also called the *voltage-to-current* map, since the function $\Lambda_{\gamma,\partial\Omega} f$ gives the induced current flux trough the boundary $\partial\Omega$. The key (*inverse*) problem is whether on can determine the conductivity matrix γ by knowing electrical boundary measurements, i.e. the corresponding Dirichlet-to-Neumann operator? In general this operator does not determine the matrix γ uniquely, see e.g. [GrUl].

The main question in this context is to find sufficient conditions insuring that the inverse problem is uniquely soluble.

2. The problem of electrical current flux in the form (**P1**) is an example of so-called *diffusive* Laplacian transport [Zag]. Besides the voltage-to-current problem the motivation to study of this kind of transport comes for instance from the transfer across *biological membranes*, see e.g. [Sap], [GrFiSap].

Let some "species" of concentration $C(p)$, $x \in \mathbb{R}^d$, diffuse stationary in the *isotropic* bulk ($\gamma = I$) from a (distant) source localised on the closed boundary $\partial\Omega$ towards a *semipermeable* compact interface ∂B of the *cell* $B \subset \Omega$, where they disappear at a given rate $W \geq 0$. Then the *steady* field of concentrations (Laplacian transport with a diffusion coefficient $D \geq 0$) obeys the set of equations:

$$(\mathbf{P2})^* \quad \begin{cases} \Delta C = 0, \ p \in \Omega \setminus \overline{B} \ , \\ C \mid_{\partial\Omega} (p) = C_0, \ \text{a constant concentration at the source } \partial\Omega \ , \\ -D \ \partial_\nu C \mid_{\partial B} (\omega) = W \ (C - C^*) \mid_{\partial B} (\omega), \ \text{on the interface } \omega \in \partial B \end{cases}$$

Usually one supposes that $C(p) = C^* \geq 0$, $p \in \overline{B}$, is a constant concentration of the "species" inside the cell $\overline{B}$.

This example motivates the following abstract *stationary* diffusive Laplacian transport problem with *absorption* on the surface ∂B:

$$(\mathbf{P2}) \qquad \begin{cases} \Delta u = 0, \ p \in \Omega \setminus \overline{B} \ , (u(p) = Const \ , \ p \in \overline{B}), \\ u \mid_{\partial\Omega} (p) = f(p), \quad p \in \partial\Omega \ , \\ (\alpha \ u + \partial_\nu u) \mid_{\partial B} (\omega) = h(\omega), \quad \omega \in \partial B \ . \end{cases}$$

This is the Dirichlet problem for domain $\Omega \supset B$ with the Dirichlet-Neumann (or Robin [Kell]) boundary condition on the absorbing surface ∂B. Varying α between $\alpha = 0$ and $\alpha = +\infty$ one recovers respectively the Neumann and the Dirichlet boundary conditions.

Now similar to (1.1) we can associate with the problem **(P2)** a Dirihlet-to-Neumann operator

$$\Lambda_{\gamma=I,\partial\Omega} : f \mapsto \partial_\nu u_f \mid_{\partial\Omega} =: g \ . \tag{1.2}$$

Domain $\mathrm{dom}(\Lambda_{I,\partial\Omega})$ belongs to a certain *Sobolev* space of functions on the boundary $\partial\Omega$, which contains $u_f := U_f^{(\alpha,h)}$, the solutions of the problem **(P2)** for given f and for the Robin boundary condition on ∂B fixed by α and h.

Then there are at least two (in fact related) *geometrical* inverse problems that make interest:

(a) Given Dirichlet data f and the corresponding (measured) Neumann data g (1.2) on the accessible *outer* boundary $\partial\Omega$, to reconstruct the shape of the interior boundary ∂B.

(b) A simpler inverse problem concerns to localisation of the domain (cell) B with a *given shape* and the fixed parameters α and h.

3. The aim of the present paper is to study the above problems (a) and (b) in the framework of application outlined in the problem **(P2)*** and to work out the corresponding formalism based on the Dirichlet-to-Neumann operators.

In Section 2.1 we formulate the mathematical setup of the these problems, and we consider uniqueness of the forward boundary value problem **(P2)** solution.

There we illustrate our strategy by an explicit example of one-dimensional inverse problem for $\Omega \subset \mathbb{R}^1$ and $B = (a, b)$.

Our main results (Section 3) concerns the two-dimensional case, when the compact $\Omega \subset \mathbb{R}^2$. Notice that there are three points that need a particular attention. The first is that the problems **(P2)*** and **(P2)** are formulated for *not* simply connected domains $\Omega \setminus \overline{B}$. The second point concerns the peculiarity of the combination of Dirichlet and Robin boundary conditions. As the third point one has to mention that the *geometrical* inverse problem is *ill-posed*.

The present paper presents first of all the *formal* solution for the case when $\alpha = +\infty$, i.e. the Dirichlet boundary conditions $u\mid_{\partial B}(\omega) = 0$, $\omega \in \partial B$. For this case our approach is motivated by important papers [LiuLu], [R]. Here we refine their results in the framework of the Dirichlet-to-Neumann formalism and add certain observations in the case of a fixed geometry of domains B and Ω following [BayZag].

In Section 4 we consider an explicit example and give numerical calculation for constant external boundary condition $f = 1$ to illustrate abstract results for $\alpha = +\infty$.

For finite $\alpha \geq 0$ and $h = 0$ we restrict ourself only by few remarks (Section 5) and we reserve the rest for the future publications. The same concerns our formal scheme for $d = 2$, since the corresponding inverse problem is *ill-posed*.

The case $d = 1$ allows explicit calculations and serves for illustration of our main ideas. Whereas for solution of the inverse problems for $d = 2$ we use a method of conformal mappings for harmonic functions in doubly connected domains $\Omega \setminus B$.

2. Setup of the Problems and Uniqueness

1. Below we suppose that Ω and $B \subset \Omega$ be open bounded domains in $\mathbb{R}^d$ with C^2-smooth disjoint boundaries $\partial\Omega$ and ∂B, that is $\partial(\Omega \setminus \overline{B}) = \partial\Omega \cup \partial B$ and $\partial\Omega \cap \partial B = \emptyset$.

Then the unit *outer-normal* to the boundary $\partial(\Omega \setminus \overline{B})$ vector-field $\nu(p)_{p \in \partial(\Omega \setminus B)}$ is well-defined, and we consider the normal derivative in **(P2)** as the *interior* limit:

$$(\partial_\nu u)\mid_{\partial B}(\omega) := \lim_{p \to \omega} \nu(\omega) \cdot (\nabla u)(p)\ ,\ p \in \Omega \setminus \overline{B}\ . \tag{2.1}$$

The existence of the limit (2.1) as well as the restriction $u\mid_{\partial B}(\omega) := \lim_{p \to \omega} u(p)$ is insured since u has to be harmonic solution of the problem **(P2)** for C^2-smooth boundaries $\partial(\Omega \setminus \overline{B})$, [Tay].

Now we introduce some indispensable standard notations and definitions [HuNa]. Let $\mathcal{H}$ be Hilbert space $L^2(M)$ on domain $M \subset \mathbb{R}^d$ and $\partial\mathcal{H} := L^2(\partial M)$ denote the corresponding *boundary space*. By $W_2^s(M)$ we denote the Sobolev space of $L^2(M)$-functions, whose s-derivatives are also in $L^2(M)$, and similar, $W_2^s(\partial M)$ is the Sobolev space of $L^2(\partial M)$-functions on the C^2-smooth boundary ∂M.

Proposition 2.1. *Let $f, h \in W_2^{1/2}(\partial\Omega)$ for C^2-smooth boundaries $\partial(\Omega \setminus B)$. If $\alpha \geq 0$, then the Dirichlet-Robin problem* **(P2)** *has a unique (harmonic) solution in domain $\Omega \setminus \overline{B}$.*

Proof. For existence we refer to [Tay]. To prove the uniqueness we consider the problem **(P2)** for $f = 0$ and $h = 0$. Then by the Gauss-Ostrogradsky theorem one gets that the corresponding solution u yields:

$$\int_{\Omega\setminus\overline{B}} dp\ (\nabla\overline{u(p)} \cdot \nabla u)(p)) = \int_{\Omega\setminus\overline{B}} dp\ \operatorname{div}(\overline{u(p)}\ (\nabla u)(p)) =$$
$$\int_{\partial B} d\sigma(\omega)\ \overline{u(\omega)}\ (\partial_\nu u)(\omega) = -\alpha \int_{\partial B} d\sigma(\omega)\ |u(\omega)|^2 \leq 0\ . \tag{2.2}$$

The estimate (2.2) implies that $u(x \in \Omega \setminus \overline{B}) = Const$. Hence by the Robin boundary condition one gets $(\alpha\, u)\ |_{\partial B}\ (\omega) = 0$, and by virtue of $u\ |_{\partial\Omega}\ (p) = f(x \in \partial\Omega) = 0$ we obtain that for $\alpha \geq 0$ the harmonic function $u(p) = 0$ for $x \in \Omega \setminus \overline{B}$. □

The next statement is a key for analysis of the inverse geometrical problems (a) and (b). Since below we use it in the case $\mathbb{R}^2$, our formulation is two-dimensional.

Proposition 2.2. *Consider two problems* **(P2)** *corresponding to a bounded domain $\Omega \subset \mathbb{R}^2$ with C^2-smooth boundary $\partial\Omega$ and to two subsets B_1 and B_2 with the same smoothness of the boundaries ∂B_1, ∂B_2. If for solutions $u_{f,h}^{(1)}$, $u_{f,h}^{(2)}$ of these problems one has*

$$\partial_\nu u_{f,h}^{(1)}\ |_{\partial\Omega} = \partial_\nu u_{f,h}^{(2)}\ |_{\partial\Omega}\ , \tag{2.3}$$

then $\partial B_1 = \partial B_2$.

Proof. By virtue of $u_{f,h}^{(1)}\ |_{\partial\Omega} = u_{f,h}^{(2)}\ |_{\partial\Omega} = f$ and by condition (2.3), the problem **(P2)** has two solutions for identical external (on $\partial\Omega$) and internal (on ∂B_1 and ∂B_2) Robin boundary conditions. Then by the standard arguments based on the Holmgren uniqueness theorem [Tat] for harmonic functions on $\mathbb{R}^2$ one obtains that $\partial B_1 = \partial B_2$. □

2. We finish this section by a simple illustration of the explicit solution of the Inverse Problems (a) and (b) in one-dimensional case. Motivated by the Laplace transport **(P2)*** we consider the case: $f = c_0$, $h = \alpha\, c^*$, and $\alpha = W/D \geq 0$, for $\Omega := (-R, R) \subset \mathbb{R}^1$ and $B := (a, b)$:

$$(\mathbf{P_{d=1}}) \qquad \begin{cases} \Delta u = 0,\ x \in (-R, R) \setminus [a, b]\ , \\ u\ |_{\partial\Omega}\ (x = \mp R) = f(\mp R) =: c_\mp, \\ (\alpha\ u + \partial_\nu u)\ |_{\partial[a,b]}\ (a) = (\alpha\ u + \partial_\nu u)\ |_{\partial[a,b]}\ (b) = \alpha\, c^*\ , \end{cases}$$

where $R > 0$ and $-R < a < b < R$.

The solution of the problem (a) is straightforward, since in the one-dimensional case the shape of absorbing cell is trivial: it is the interval $B := (a, b)$.

Now notice that a general solution of the problem $(\mathbf{P_{d=1}})$ is a combination of linear functions supported in domain $\Omega := (-R, R) \setminus [a, b]$ and a constant c^* in the interval $[a,b]$:

$$-R < x < a: \qquad u(x) = -\frac{c_- - c^*}{(R+a)+\alpha^{-1}}(R+x) + c_- \ , \tag{2.4}$$
$$a \le x \le b: \qquad u(x) = c^* \ ,$$
$$b < x < R: \qquad u(x) = -\frac{c_+ - c^*}{(R-b)+\alpha^{-1}}(R-x) + c_+ \ . \tag{2.5}$$

Given Dirichlet data c_0 on the boundary $\partial\Omega$ and *measuring* on this boundary the Neumann data in the form of the flux currents:

$$j_- := -\partial_\nu u\mid_{\partial\Omega} (x=-R) = \frac{c_- - c^*}{(R+a)+\alpha^{-1}}$$
$$j_+ := -\partial_\nu u\mid_{\partial\Omega} (x=+R) = -\frac{c_+ - c^*}{(R-b)+\alpha^{-1}}$$

one can solve explicitly the both problems (a) and (b).

In the one-dimensional case the *shape* of the cell is defined by its *size*: $(b-a)$, whereas localization is fixed by the points:

$$a = (c_- - c^*)/j_- - R - \alpha^{-1} \ ,$$
$$b = (c_+ - c^*)/j_+ + R + \alpha^{-1} \ .$$

3. Two-Dimensional Inverse Problem: Conformal Mapping and the Shape of ∂B

1. The relevance of the conformal mapping in the study of the boundary value problems for harmonic functions (solutions of the Laplace equation) is well-known, see e.g. [LaCh] (Ch.III), or [Mih] (Ch.13).

Recall that if the complex function $w : z \mapsto \mathbb{C}$ is holomorphic in the open domain $\{\Omega \subset \mathbb{C} : z = x + i\,y \in \Omega\}$, then by the Cauchy-Riemann conditions the functions $u(x,y) := (\mathrm{Re}\,w)(x,y)$ and $v(x,y) := (\mathrm{Im}\,w)(x,y)$ are harmonic in Ω. Here $w(z) = u(x,y) + i\,v(x,y)$.

Remark 3.1. *There is an elementary inverse problem of the complex analysis : given a harmonic function $u(x,y)$ in Ω to construct in this domain the harmonic function $v(x,y)$ (harmonic conjugate to u) such that the complex function $w = u + i\,v$ is holomorphic. In fact one finds the harmonic conjugate from the Cauchy-Riemann conditions*

$$\partial_x u = \partial_y v \quad , \quad \partial_y u = -\partial_x v \ , \tag{3.1}$$

since for a given u this is a system of partial differential equations for v. Notice that for a simply connected domain Ω the solution of this system always exists and it is unique up to a constant, whereas in non-simply connected domains the harmonic conjugate may not be a single-valued function. On the other hand, in any simply connected subset $\Omega_0 \subset \Omega$ one can select a single-valued branch of this function. By consequence this means a selection of the single-valued branch of the total complex function w.

Application of conformal mappings to analysis of harmonic functions and Laplace equation is based on the following observations:

Proposition 3.2. *Let $\zeta : z \mapsto \zeta(z)$ be a conformal mapping $\zeta(z) : N \to M$ by a holomorphic function $\zeta(z) = \xi(x,y) + i\eta(x,y)$. If the function $\widetilde{u}(\xi,\eta)$ is harmonic in M, then the composition*

$$u(x,y) := (\widetilde{u} \circ \zeta)(x,y) = \widetilde{u}(\xi(x,y), \eta(x,y)) \ , \tag{3.2}$$

is a harmonic function of x, y in N.

In particular one obtains:

$$(\Delta_z u)(x,y) = |\partial_z \zeta(z)|^2 \ (\Delta_\zeta \widetilde{u})(\xi(x,y), \eta(x,y)) \ . \tag{3.3}$$

(Here we explicitly distinguish Laplacians in different coordinates, $\Delta_z := \partial_x^2 + \partial_y^2$ and $\Delta_\zeta := \partial_\xi^2 + \partial_\eta^2$, but we ignore these subindexes below, since it will not produce any confusion.) Notice that this statement is based only on a straight forward application of the Cauchy-Riemann conditions for the mapping $\zeta(z)$, i.e. it *does not* assume the existence of a harmonic conjugate neither for $\widetilde{u}$, nor for u. Although for a simply connected $N_0 \subset N$ one can show that every harmonic function is a *real part* of a branch of holomorphic in N_0 function.

The second observation is related to the Dirichlet-to-Neumann formalism and makes clear the importance of the notion of the *harmonic conjugate* function, [LaCh], Ch.III.

Proposition 3.3. *Let Ω be open simply connected bounded domain in $\mathbb{R}^2$ with a C^2-smooth boundary $\partial\Omega$. Then solution of the Neumann problem*

$$(\mathbf{P_N}) \qquad \begin{cases} \Delta u = 0, \ p \in \Omega \setminus \overline{B} \ , \\ \partial_\nu u \,|_{\partial\Omega} (p) = g(p), \quad p \in \partial\Omega \ . \end{cases}$$

reduces to the Dirichet problem for the function v, which is harmonic conjugate to the function u.

To make this evident, notice first that the normal derivative here is defined in the sense of (2.1). Let the boundary $\partial\Omega$ be parameterized by the natural parameter of its arc-length: $\partial\Omega = \{\Gamma(\tau) \in \mathbb{C}\}_{\tau \in [0,l)}$. Then the Cauchy-Riemann conditions (3.1) imply that

$$\partial_\tau v \,|_{\partial\Omega} (p) = \partial_\nu u \,|_{\partial\Omega} (p) = g(p). \tag{3.4}$$

Since by integration along the contour Γ one obtains

$$v(p_1) = v(p_0) + \int_{\tau_0}^{\tau_1} d\tau \ \partial_\tau v(\Gamma(\tau)) = v(p_0) + \int_{\tau_0}^{\tau_1} ds \ g(\Gamma(\tau)) =: f(p_1) \ ,$$

the solution of $(\mathbf{P_N})$ is equivalent to the Dirichlet problem $(\mathbf{P_D})$ for v and the boundary condition f.

2. To outline the main steps in reconstructing the unknown boundary ∂B, we consider first the problem **(P2)** for the Dirichlet case $\alpha = +\infty$:

$$(\mathbf{P^{\infty}_{d=2}}) \qquad \begin{cases} \Delta u = 0, \ p \in \Omega \setminus \overline{B} \ , \\ u \mid_{\partial\Omega} (p) = f(p), \quad p \in \partial\Omega \ , \\ u \mid_{\partial B} (\omega) = 0, \quad \omega \in \partial B \ . \end{cases}$$

It is well-known, see e.g. [LaCh], [Mar], that the doubly connected bounded domain $\Omega \setminus \overline{B}$ is the image of a conformal mapping of an annulus

$$A_B := \{z \in \mathbb{C} : 0 < \rho_B < |z| < 1\} \tag{3.5}$$

produced by a bijective holomorphic function $\zeta(z)$. This function maps boundaries to boundaries: $\zeta : C_{\rho_B} \to \partial B$ and $\zeta : C_{r=1} \to \partial\Omega$.

(i) The first step is to find the trace $\zeta \mid_{C_1}$ of the unknown function $\zeta(z)$ on the external unit circle $C_{r=1}$.

(ii) Then the next step is to reconstruct the function $\zeta(z)$ in the whole annulus A_B, which solves the geometrical inverse problem (see Introduction 1.2 (a)) by tracing the boundary ∂B as the limit of ζ from inside: $\partial B = \{\zeta(z)\} \mid_{z \to C_{\rho_B}} := \zeta(C_{\rho_B})$.

(i) Let external boundary in the problem $(\mathbf{P^{\infty}_{d=2}})$ be parameterized by the natural parameter of its arc-length: $\partial\Omega = \{\Gamma(\tau) \in \mathbb{C}\}_{\tau \in [0,l)}$. Then the trace of the conformal mapping $\zeta : C_1 \to \partial\Omega$ defines by the equation:

$$\zeta(e^{i\phi}) = \Gamma(\tau) \quad , \quad \text{for } \phi \in [0, 2\pi) \ , \tag{3.6}$$

with the condition $\zeta(e^{i\phi}) \mid_{\phi=0} = \Gamma(0)$, a bijective function $\phi : \tau \mapsto \phi(\tau) \in [0, 2\pi)$.

Therefore, to calculate the trace of the function $\zeta(z)$ on the external unit circle $C_{r=1}$ is equivalent to find a solution $\phi(\tau)$ of (3.6), or the corresponding inverse function $\tau(\phi)$.

To this end, let u_f be solution of the problem $(\mathbf{P^{\infty}_{d=2}})$. Then by Proposition 3.2 the function $\widetilde{u}_{\widetilde{f}} := u_f \circ \zeta$ is harmonic in the annulus A_B and it is solution of the Dirichlet problem

$$(\widetilde{\mathbf{P}}^{\infty}_{\mathbf{d=2}}) \qquad \begin{cases} \Delta \widetilde{u} = 0, \ p \in A_B \ , \\ \widetilde{u} \mid_{C_1} (p) = \widetilde{f}(p) \ , \quad p \in C_1 \ , \\ \widetilde{u} \mid_{C_{\rho_B}} (\omega) = 0, \quad \omega \in C_{\rho_B} \ . \end{cases}$$

Here $\widetilde{f}(p) = (f \circ \zeta)(p) = f(\zeta(p)) = f(\xi(x,y), \eta(x,y))$ and $p = (x,y) \in C_1$.

Consider the solution u_f of the Dirichlet problem $(\mathbf{P^{\infty}_{d=2}})$. Then the Dirichlet-to-Neumann operator $\Lambda_{\partial\Omega}$ for the external boundary $\partial\Omega$ is defined similar to (1.2):

$$\Lambda_{\partial\Omega} f = \partial_\nu u_f \mid_{\partial\Omega} =: g \ . \tag{3.7}$$

Let v_f be harmonic conjugate to u_f. Then by (3.4) we obtain that for external boundary $\partial\Omega$

$$\begin{aligned} &\partial_\tau v_f \mid_{\partial\Omega} (\tau) = \partial_\tau v_f(\Gamma(\tau)) = \partial_\nu u_f(\Gamma(\tau)) = (\Lambda_{\partial\Omega} f)(\Gamma(\tau)) = \\ &(\Lambda_{\partial\Omega} f)(\zeta(e^{i\phi(\tau)})) = (\Lambda_{\partial\Omega} f \circ \zeta)(e^{i\phi(\tau)}) \ . \end{aligned} \tag{3.8}$$

With conformal mapping ζ the relation (3.8) can be rewritten as:

$$\partial_\tau v_f(\Gamma(\tau)) = \partial_\tau v_f(\zeta(e^{i\phi(\tau)})) = \partial_\phi(v_f \circ \zeta)(e^{i\phi(\tau)}))\partial_\tau\phi(\tau) \ . \tag{3.9}$$

Since $\widetilde{u}_{\widetilde{f}} := u_f \circ \zeta$ and $\widetilde{v}_{\widetilde{f}} := v_f \circ \zeta$, see $(\widetilde{\mathbf{P}}^{\infty}_{\mathbf{d=2}})$, by (3.4), we obtain that

$$\partial_\phi(v_f \circ \zeta)(e^{i\phi})) = \partial_\phi \widetilde{v}_{\widetilde{f}}(\phi) = \partial_\nu \widetilde{u}_{\widetilde{f}} \mid_{C_1} (\phi) = \Lambda_{C_1}(f \circ \zeta)(e^{i\phi}) \ , \tag{3.10}$$

with a usual convention about the normal derivative $\partial_\nu(\cdot) \mid_{C_1}$ on the unit circle C_1. Here $\Lambda_{C_1} : \widetilde{f} \mapsto \partial_\nu \widetilde{u}_{\widetilde{f}} \mid_{C_1}$ is the Dirichlet-to-Neumann operator corresponding to the problem $(\widetilde{\mathbf{P}}^{\infty}_{\mathbf{d=2}})$.

Relations (3.8)-(3.10) yield differential equation for $\phi = \phi(\tau)$:

$$\partial_\tau \phi = \frac{(\Lambda_{\partial\Omega} f \circ \zeta)(e^{i\phi})}{\Lambda_{C_1}(f \circ \zeta)(e^{i\phi})} \ . \tag{3.11}$$

For a given boundary Γ the solution $\phi(\tau)$ of equation (3.11) gives the trace of the function $\zeta(z)$ on the circle C_1. Indeed, by (3.6) we obtain that on C_1 it is defined by

$$\zeta(e^{i\phi}) = \Gamma(\tau(\phi)) \quad , \quad \text{for } \phi \in [0, 2\pi) \ , \tag{3.12}$$

where $\tau(\phi)$ is the function, which is inverse to $\phi(\tau)$.

3. Hence, for a fixed boundary Γ one can in principle find the trace $\zeta(z) \mid_{C_1}$ using the scheme outlined above. To this end let $\widetilde{f} \in W_2^1(C_1)$, where we identify C_1 with $[0, 2\pi]$, see problem $(\widetilde{\mathbf{P}}^{\infty}_{\mathbf{d=2}})$. Then solution of this problem gets the form:

$$\widetilde{u}_{\widetilde{f}}(\rho, \phi) = a_0 \ \ln\rho + b_0 + \tag{3.13}$$
$$\sum_{n=1}^{\infty} \left[(a_n \rho^n + b_n \rho^{-n}) \cos n\phi + (c_n \rho^n + d_n \rho^{-n}) \sin n\phi\right] \ ,$$

The coefficients in expansion (3.13) are equal to

$$a_n = \frac{\widetilde{f}_{1,n}}{(1 - \rho_B^{2n})} \ , \ b_n = -\frac{\rho_B^{2n} \ \widetilde{f}_{1,n}}{(1 - \rho_B^{2n})} \ , \ a_0 = -\frac{\widetilde{f}_{1,0}}{\ln \rho_B} \ , \quad b_0 = \widetilde{f}_{1,0} \ , \tag{3.14}$$

$$c_n = \frac{\widetilde{f}_{2,n}}{(1 - \rho_B^{2n})} \ , \ d_n = -\frac{\rho_B^{2n} \ \widetilde{f}_{2,n}}{(1 - \rho_B^{2n})} \ . \tag{3.15}$$

They are related to coefficients of the Fourier series for $\widetilde{f}(\phi)$:

$$\widetilde{f}_{1,0} = \frac{1}{2\pi} \int_0^{2\pi} d\phi \widetilde{f}(\phi), \ \widetilde{f}_{1,n} = \frac{1}{\pi} \int_0^{2\pi} d\phi \widetilde{f}(\phi) \cos n\phi, \ \widetilde{f}_{2,n} = \frac{1}{\pi} \int_0^{2\pi} d\phi \widetilde{f}(\phi) \sin n\phi.$$

Then the corresponding Dirichlet-to-Neumann operator (3.10) acts as a bounded operator from $W_2^1(C_1)$ to $L^2(C_1)$:

$$\Lambda_{C_1} \widetilde{f}(\phi) = \partial_\nu \widetilde{u}_{\widetilde{f}} \mid_{C_1} (\phi) = \tag{3.16}$$
$$-\frac{\widetilde{f}_{1,0}}{\ln \rho_B} + \sum_{n=1}^{\infty} n \left[(a_n - b_n) \cos n\phi + (c_n - d_n) \sin n\phi\right] \ .$$

By (3.10) and (3.16) we obtain the identity

$$\int_0^{2\pi} d\phi \Lambda_{C_1} \widetilde{f}(\phi) = -\frac{1}{\ln \rho_B} \int_0^{2\pi} d\phi \widetilde{f}(\phi) \ ,$$

which implies by (3.8)-(3.10) that the radius of the *internal* circle is defined as

$$\begin{aligned} \rho_B &= \exp\left\{ -\left(\int_0^{2\pi} d\phi \, (f \circ \zeta)(e^{i\phi}) \right) \left(\int_{\partial\Omega} d\tau \, \partial_\tau \phi(\tau) \partial_\phi (v_f \circ \zeta)(e^{i\phi(\tau)}) \right)^{-1} \right\} \\ &= \exp\left\{ -\left(\int_0^{2\pi} d\phi \, (f \circ \zeta)(e^{i\phi}) \right) \left(\int_{\partial\Omega} d\tau \, \partial_\nu u_f(\Gamma(\tau)) \right)^{-1} \right\} \ . \end{aligned} \tag{3.17}$$

The relation (3.17) allows to calculate ρ_B if one knows the trace $\zeta(z) \mid_{C_1}$, but since by (3.12) we have $\zeta(e^{i\phi(\tau)}) = \Gamma(\tau)$, the first equation to solve is (3.11). Notice that by definition $\partial\Omega = \{\Gamma(\tau) \in \mathbb{C}\}_{\tau \in [0,l)}$ and by (3.6),(3.11) one gets that there is a constraint:

$$l = \int_0^{2\pi} d\phi \, \frac{\Lambda_{C_{\rho_B}, C_1} (f \circ \zeta)(e^{i\phi})}{(\Lambda_{\partial B, \partial\Omega} f \circ \zeta)(e^{i\phi})} \ , \tag{3.18}$$

as well as that solution $\tau(\phi)$ of (3.11) must be a 2π-periodic function of ϕ. Here we explicitly recall the second boundary dependence for the both Dirichlet-to-Neumann operators: $\Lambda_{C_1} = \Lambda_{C_{\rho_B}, C_{\rho=1}}$ and $\Lambda_{\partial\Omega} = \Lambda_{\partial B, \partial\Omega}$.

Example 3.4. *We illustrate the above by a trivial example of the round Dirichlet absorbing cell. Let boundaries $\partial\Omega = C_R$ and $\partial B = C_{r_B}$ be two concentric circles with radius r_B, which is the only unknown parameter that should be defined as a solution of the inverse geometrical problem.*

Following our scheme the domain $\Omega \setminus \overline{B}$ is the image of a conformal mapping of an annulus

$$A_B := \{ z \in \mathbb{C} : 0 < \rho_B < |z| < 1 \} \tag{3.19}$$

produced by a bijective holomorphic function $\zeta(z)$. This function maps boundaries to boundaries: $\zeta : C_{\rho_B} \to \partial B$ and $\zeta : C_{r=1} \to \partial\Omega$.

By virtue of the rotational symmetry one can try to solve this problem for ∂B via $(\mathbf{P}^{\infty}_{\mathbf{d=2}})$ with boundary condition $u \mid_{\partial\Omega} (p) = f$ independent of $\arg(p)$. Then solution of the direct problem $(\mathbf{P}^{\infty}_{\mathbf{d=2}})$ is given by the $n = 0$ version of (3.13): $u_f(\rho, \phi) = a \ \ln \rho + b$ for $r_B < \rho < R$. Taking into account boundary conditions one finds a and b and the explicit form of the corresponding Dirichlet-to-Neumann operator, cf (3.7):

$$\Lambda_{\partial B, \partial\Omega} : f \mapsto \partial_\nu u_f \mid_{C_R} = \frac{1}{R \, (\ln R - \ln r_B)} \, f =: g \ . \tag{3.20}$$

(*Note that our example is so simple that the one-measure of the "voltage-current" couple $\{f, J\}$, where $\{J := \Lambda_{\partial B, \partial\Omega} f = g\}$, is enough to define uniquely the operator $\Lambda_{\partial B, \partial\Omega}$ that solves the problem of r_B explicitly.*)
Since the conformal mapping gives for the exterior boundaries gives $\zeta(e^{i\phi}) = R \ e^{i\phi}$, for $p \in C_1$ (trace $\zeta \mid_{C_1}$), one gets $\widetilde{f}(p) := (f \circ \zeta)(p) = f(\zeta(e^{i\phi})) =$

$f(Re^{i\phi}) = f$. Then by (3.13) the Dirichlet-to-Neumann operator for the problem $(\widetilde{\mathbf{P}}^{\infty}_{\mathbf{d=2}})$ has the form:

$$\Lambda_{C_{\rho_B},C_{\rho=1}} : \widetilde{f} \mapsto \partial_\nu \widetilde{u}_{\widetilde{f}} \,|_{C_1} = -\ \frac{1}{\ln \rho_B}\ \widetilde{f}\ . \tag{3.21}$$

Then by (3.20) we get for numerator in (3.11):

$$(\Lambda_{\partial B,\partial\Omega} f \circ \zeta) = \frac{1}{R\ (\ln R - \ln r_B)}\ f \circ \zeta = \left\{ R\ \ln \frac{R}{r_B} \right\}^{-1} \widetilde{f}, \tag{3.22}$$

and by (3.21) one obtains for denominator in (3.11):

$$\Lambda_{C_{\rho_B},C_{\rho=1}}(f \circ \zeta) = -\ \frac{1}{\ln \rho_B}\ \widetilde{f}\ . \tag{3.23}$$

Inserting (3.22) and (3.23) into (3.17) (or into (3.18), where $l = 2\pi R$) we obtain that $\rho_B = r_B/R$, i.e. for internal boundaries the conformal mapping gives: $\zeta(\rho_B e^{i\phi}) = r_B e^{i\phi} = R\ \rho_B e^{i\phi}$. This implies that the mapping is $\zeta(z) = R\ z$ (see (ii)), and also the evident final result about the form of the boundary ∂B as the trace of $\zeta(z)$ on the C_{ρ_B}.

4. This example shows that $\tau(\phi)$ is a 2π-periodic extension of the *linear* function

$$\tau_0(\phi) := \frac{l}{2\pi}\,\phi\ ,\quad \phi \in [0, 2\pi)\ . \tag{3.24}$$

The reason is a simple linear form of the corresponding conformal mapping. Any deviation from the concentric domains $\partial\Omega = C_R$ and $\partial B = C_{r_B}$ makes the function $\tau(\phi)$ non-linear, but obeying the condition (3.18).

As a less trivial application of the scheme presented above is the example of non-concentric domains $\partial\Omega = C_R$ and $\partial B = C_{r_B}$. In this case the conformal mapping ζ is *a priori* known: it is the Möbius transformation, and one can proceed with this trial ζ along the same line of reasoning as in Example 3.4, see [BayZag]. Although illustration of the inverse geometrical problem solution needs a complete application of the above formalism, since now one has to solve two coupled equations (3.11) and (3.17) with condition (3.18).

(ii) We rewrite these equations (incorporating the constraint (3.18)) in the following form:

$$\rho_B = \exp\left\{ -\left(\int_0^{2\pi} d\phi\,(f \circ \zeta)(e^{i\phi}) \right) \left(\int_{\partial\Omega} d\tau\, \partial_\nu u_f(\Gamma(\tau)) \right)^{-1} \right\}. \tag{3.25}$$

$$\partial_\phi \tau = \frac{l}{2\pi} + \frac{\Lambda_{C_{\rho_B},C_1}(f \circ \zeta)}{(\Lambda_{\partial B,\partial\Omega} f \circ \zeta)} - \frac{1}{2\pi} \int_0^{2\pi} d\phi\, \frac{\Lambda_{C_{\rho_B},C_1}(f \circ \zeta)(e^{i\phi})}{(\Lambda_{\partial B,\partial\Omega} f \circ \zeta)(e^{i\phi})}\ . \tag{3.26}$$

Notice that by (3.22) and (3.23) for concentric domains $\partial\Omega = C_R$ and $\partial B = C_{r_B}$ the last two terms in (3.26) cancels. Therefore, one can consider this case as the *zero-order* approximation $\tau = \tau_0(\phi)$ for the solution of (3.26) with $\zeta = \zeta_0(z) := z$ and $\rho_B = \rho_0 := r_B/R$. This observation inspires to consider equations (3.25) and (3.26), together with relations $\zeta_n(e^{i\phi}) = \Gamma(\tau_n(\phi))$, see (3.12), as a non-linear iterative scheme to obtain ρ_B and the function $\tau(\phi)$ (or $\zeta(z)$), cf [LiuLu]:

$$\begin{aligned}
\rho_n &= \exp\left\{-\left[\int_0^{2\pi} d\phi (f\circ\zeta_n)(e^{i\phi})\right]\left[\int_{\partial\Omega} d\tau \partial_\nu u_f(\Gamma(\tau))\right]^{-1}\right\}, && (3.27)\\
\partial_\phi \tau_{n+1} &= \frac{l}{2\pi} + \frac{\Lambda_{C_{\rho_n},C_1}(f\circ\zeta_n)}{(\Lambda_{\partial B,\partial\Omega} f\circ\zeta_n)} - \frac{1}{2\pi}\int_0^{2\pi} d\phi\, \frac{\Lambda_{C_{\rho_n},C_1}(f\circ\zeta_n)(e^{i\phi})}{(\Lambda_{\partial B,\partial\Omega} f\circ\zeta_n)(e^{i\phi})}, && (3.28)\\
\zeta_n(e^{i\phi}) &= \Gamma(\tau_n(\phi))\ . && (3.29)
\end{aligned}$$

Remark 3.5. *Suppose that for $n \to \infty$ the iterations converge: $\rho_n \to \rho_B$, $\tau_n(\phi) \to \tau(\phi)$ and for given Γ: $\zeta_n(z) \to \zeta(z)$. Then the function $\Gamma(\tau(\phi))$ can be presented as the Fourier series:*

$$\Gamma(\tau(\phi)) = \sum_{s\in\mathbb{Z}} \gamma_s e^{is\phi}\ . \qquad (3.30)$$

Since $\Gamma(\tau(\phi))$ is the image of the external boundary C_1 by the seeking function $\zeta(z)$, the coefficients γ_s are the same as in the Laurent series for this function in the annulus A_B:

$$\zeta(z) = \sum_{s\in\mathbb{Z}} \gamma_s z^s\ . \qquad (3.31)$$

Now the final step is to observe that the unknown internal boundary ∂B coincides with the conformal image $\{\Gamma_{\partial B}(\phi)\}_{0\leq\phi<2\pi} = \zeta(C_{\rho_B})$ of the internal A_B-circle C_{ρ_B} with the radius $\rho_B < 1$ calculated by iterations (3.27):

$$\Gamma_{\partial B}(\phi) = \sum_{s\in\mathbb{Z}} (\rho_B)^s\ \gamma_s e^{is\phi}\ . \qquad (3.32)$$

The relation (3.32) solves *formally* the inverse geometrical problem for Dirichlet boundary conditions on the unknown contour $\partial B = \{\Gamma_{\partial B}(\phi)\}_{0\leq\phi<2\pi}$.

4. Example: Constant boundary conditions

1.1 Problem $(P_{f_\pm=1,0})$**.** Below we suppose that Ω and $B \subset \Omega$ be open bounded domains in $\mathbb{R}^2$ with C^2-smooth disjoint boundaries $\partial\Omega$ and ∂B, that is $\partial(\Omega\setminus\overline{B}) = \partial\Omega \cup \partial B$ and $\partial\Omega \cap \partial B = \emptyset$.

The *unknown internal boundary* ∂B should be find from the solution u of the Dirichlet problem:

$$(\mathbf{P_{f_\pm=1,0}}) \qquad \begin{cases} \Delta u = 0,\ p \in \Omega\setminus\overline{B}\ , \\ u\mid_{\partial\Omega}(p) = f_+(p) = 1,\quad p\in\partial\Omega\ , \\ u\mid_{\partial B}(p) = f_-(p) = 0,\quad p\in\partial B\ , \end{cases}$$

with help of the *given* (measured) Neumann *data*: $g(p) = \partial_\nu u\mid_{\partial\Omega}(p)$, *exterior* normal derivative on the external boundary $p \in \partial\Omega$.

Remark 4.1. *Notice that one can always find a conformal mapping that transforms domain Ω into unit disc. Therefore, we put for simplicity $\Omega = D_{r=1}$, the unit disc, i.e. $\partial\Omega = C_1$, is the unit circle.*

Remark 4.2. *Since below we use a conformal map approach to localisation the internal boundary ∂B, we identify the $\mathbb{R}^2$-points $p = (x, y)$ with those of the complex plane $\mathbb{C}$ by: $p \mapsto z(p) := x + iy \in \mathbb{C}$. Then it is known, see e.g.* [LaCh], *that the harmonic function solving $(\mathbf{P_{f_\pm=1,0}})$ can be viewed as the real part of a holomorphic in domain $\Omega \setminus \overline{B}$ function $\widehat{u}(z)$, i.e., $u(p) = \mathrm{Re}\ \widehat{u}(z(p))$. We put $\widehat{u}(z) = u(x, y) + iv(x, y)$, where $v(x, y)$ is harmonic conjugate to $u(x, y)$,* [LaCh]. *Recall that for a double connected domain the function $\widehat{u}(z)$ may be multi-valued. Then we consider for $\widehat{u}(z)$ only one (principle) branch.*

Remark 4.3. *Recall that in polar coordinates $z = re^{i\phi} \in \mathbb{C}$ the measured Neumann data g on C_1 take the form*

$$\begin{aligned} g(\phi) &= e_r \cdot \nabla u \mid_{z\in C_1} = (\cos\phi\ \partial_x u + \sin\phi\ \partial_y u) \mid_{z\in C_1} = \\ &= \partial_r u(r\cos\phi, r\sin\phi) \mid_{r=1} \quad . \end{aligned} \tag{4.1}$$

We also recall that the Cauchy-Riemann conditions in these coordinates can be written as

$$\partial_r u = \frac{1}{r}\ \partial_\phi v \quad , \quad \frac{1}{r}\ \partial_\phi u = -\,\partial_r v \ . \tag{4.2}$$

1.2 Problem $(P^*_{f_\pm=1,0})$**.** Let holomorphic function $w : z = (x+iy) \mapsto (w_1 + iw_2)$ maps doubly connected bounded domain $D_1 \setminus \overline{B} \subset \mathbb{C}$ into annulus

$$A_B := \{w \in \mathbb{C} : 0 < \rho_B < |w| < 1\} \ . \tag{4.3}$$

This function maps boundaries to boundaries: $w : \partial B \to C_{\rho_B}$ and $w : \partial\Omega = C_1 \to C_1$ and define the function $U(w_1, w_2)$ by

$$u(x, y) = (U \circ w)(x, y) = U(w_1(x, y), w_2(x, y)) \ . \tag{4.4}$$

Then the problem $(P_{f_\pm=1,0})$ transfers into

$$(\mathbf{P^*_{f_\pm=1,0}}) \qquad \begin{cases} \Delta U = 0,\ p \in D_1 \setminus \overline{D_{\rho_B}} \ , \\ U \mid_{C_1} (p) = 1, \quad p \in C_1 \ , \\ U \mid_{C_{\rho_B}} (p) = 0, \quad p \in C_{\rho_B} \ , \end{cases}$$

with the exterior normal derivative:

$$\partial_\nu U \mid_{z\in C_1} (w(z)) = \left(\frac{1}{|w'(z)|}\ g(z)\right)\Big|_{z\in C_1} \ . \tag{4.5}$$

Notice that the value of the normal derivative (4.5) is B-dependent via conformal mapping w.

1.3 Solution of the Problem $(P^*_{f_\pm=1,0})$**.** For the general solution one easily finds a representation in the (complex) polar coordinates $w = \rho ei\varphi$:

$$U(\rho, \varphi) = a + b\ln\rho + \sum_{n\in\mathbb{Z}\setminus 0} (a_n\rho^n e^{in\varphi} + b_n\rho^{-n}e^{-in\varphi}) \ ,$$

which is nothing but the standard Fourier-series representation. By virtue of the boundary conditions we obtain

$$a = 1 \ , \ b = -\frac{1}{\ln \rho_B} \ , \ a_n = b_n = 0 \ .$$

Then, consequently, we get for the solution the explicit form:

$$U(w_1, w_2) = U(\rho, \varphi) = \frac{\ln(\rho/\rho_B)}{\ln(1/\rho_B)} = \frac{1}{\ln(1/\rho_B)} \ \ln \frac{|w|}{\rho_B} \ , \tag{4.6}$$

and the corresponding B-dependent normal derivative on the external boundary C_1, cf. (4.5):

$$\partial_\nu U \mid_{C_1} (w) = \partial_\rho U(\rho, \varphi) \mid_{\rho=1} = \frac{1}{\ln(1/\rho_B)} \ . \tag{4.7}$$

Notice that in contrast to the Problem $(P_{f_\pm=1,0})$, the Neumann data (4.7) for the Problem $(P^*_{f_\pm=1,0})$ are *isotropic* and they depend on B only via radius ρ_B.

It is clear that to proceed with localisation of the internal boundary ∂B one has to find the conformal mapping $w(z)$. The relations (4.5) and (4.7) yield a functional equation

$$\frac{1}{\ln(1/\rho_B)} = \left(\frac{1}{|w'(z)|} \ g(z) \right) \Big|_{z \in C_1} \tag{4.8}$$

for w. This equation is insufficient, since it is localised only on the boundary C_1. To overcome this difficulty we use complex extensions of $(P_{f_\pm=1,0})$ and $(P^*_{f_\pm=1,0})$ indicated in Remark 4.2.

2.1 Complex extension. Let us define the complex extension of (4.6) by

$$\widehat{U}(w = w_1 + iw_2) := \frac{1}{\ln(1/\rho_B)} \ln \frac{w}{\rho_B} = (U + iV)(w) \ , \tag{4.9}$$

where $V = \arg w$ is harmonic conjugate to $U = \ln |w|$ and corresponds to the principle branch of the logarithm. Hence, one can similarly introduce the function

$$\widehat{u}(z) := \widehat{U}(w(z)) = (u + iv)(z) = \frac{1}{\ln(1/\rho_B)} \ln \frac{w(z)}{\rho_B} \ , \tag{4.10}$$

where v is harmonic conjugate to u.

2.2 Complex extension and the Problem $(P_{f_\pm=1,0})$**.** By (4.10) one gets

$$u(x, y) = \mathrm{Re} \ \widehat{u}(z) = \frac{1}{\ln(1/\rho_B)} \ln \frac{|w(z)|}{\rho_B} \ .$$

Let $z = re^{i\phi}$. Then by virtue of (4.1), (4.10) and by

$$\partial_r \widehat{u}(z) = (\partial_r u + i\partial_r v)(z) = \widehat{u}'(z) \, e^{i\phi} = \frac{1}{\ln(1/\rho_B)} \frac{w'(z)}{w(z)} \ e^{i\phi} \ , \tag{4.11}$$

we obtain the following equation:

$$\partial_r u \mid_{C_1} = \mathrm{Re} \left\{ \frac{1}{\ln(1/\rho_B)} \frac{w'(e^{i\phi})}{w(e^{i\phi})} \ e^{i\phi} \right\} = g(\phi) \ . \tag{4.12}$$

Notice that the Cauchy-Riemann conditions (4.2) implies

$$\partial_r v(z = re^{i\phi}) = -\frac{1}{r}\ \partial_\phi u(re^{i\phi}) = -\frac{1}{r\ \ln(1/\rho_B)}\ \partial_\phi \ln|w(re^{i\phi})|\ . \tag{4.13}$$

Since for $r = 1$ we have $|w(e^{i\phi})| = 1$, one gets $\partial_r v(z)\ |_{C_1} = 0$, i.e. the condition *Re* in (4.12) is superfluous as soon as we stick to the external boundary C_1:

$$\frac{1}{\ln(1/\rho_B)} \frac{w'(e^{i\phi})}{w(e^{i\phi})}\ e^{i\phi} = g(\phi)\ . \tag{4.14}$$

2.3 Solution for conformal mapping $w(z)$**.** Motivated by (4.14) we define a continuation of (4.12) from the external boundary C_1 into domain $\Omega \setminus \overline{B}$. To this end we introduce a holomorphic in $\Omega \setminus \overline{B}$ function F with the corresponding Laurent series:

$$F(z) := \frac{1}{\ln(1/\rho_B)} \frac{w'(z)}{w(z)}\ z = F_0 + \sum_{n=1}^{\infty}(F_n\ z^n + F_{-n}\ z^{-n})\ . \tag{4.15}$$

Then by periodicity of g and by (4.14), (4.15) we obtain the relation

$$g(\phi) = \sum_{n\in\mathbb{Z}} g_n e^{in\phi} = F(z = e^{i\phi})\ , \tag{4.16}$$

which implies $F_n = g_n$ and $\overline{g_n} = g_{-n}$, for $n \in \mathbb{Z}$, as well as equation

$$\frac{1}{\ln(1/\rho_B)} \frac{w'(z)}{w(z)}\ z = g_0 + \sum_{n=1}^{\infty}(g_n z^n + g_{-n} z^{-n})\ . \tag{4.17}$$

Therefore, one has

$$\partial_z \ln w(z) = \ln(1/\rho_B) \left[\frac{g_0}{z} + \sum_{n=1}^{\infty}(g_n z^{n-1} + g_{-n} z^{-n-1})\right]\ . \tag{4.18}$$

Hence, we obtain that

$$w(z) = w_0\ z^{g_0 \ln(1/\rho_B)} \exp\left[\ln(1/\rho_B) \sum_{n=1}^{\infty}(g_n z^n - g_{-n} z^{-n})/n\right]\ . \tag{4.19}$$

Since $w : C_1 \to C_1$, one obviously gets

$$w(e^{i\phi}) = e^{i\varphi(\phi)} \quad \text{and} \quad w(e^{i(\phi+2\pi)}) = e^{i\varphi(\phi+2\pi)} = e^{i\varphi(\phi)}\ , \tag{4.20}$$

that implies $g_0 \ln(1/\rho_B) = 1$ and

$$\rho_B = e^{-1/g_0}\ , \tag{4.21}$$

i.e., we *must* put $g_0 > 0$. Notice that $|w(e^{i\phi})| = 1$ and (4.21) yield $|w_0| = 1$, which we can choose to be real. Therefore, finally one obtains for the conformal mapping w the expression:

$$w(z) = z\ \exp\left[(1/g_0) \sum_{n=1}^{\infty}(g_n z^n - g_{-n} z^{-n})/n\right]\ , \tag{4.22}$$

which is completely defined by the measured Neumann data $g(p)$ on the external boundary C_1.

Remark 4.4. *In spite of the obvious remark: $\partial_\phi |w(e^{i\phi})| = 0$, which we used to establish (4.14), the derivative $\partial_\phi w(e^{i\phi}) = e^{i\varphi(\phi)}\,\partial_\phi \varphi(\phi) \neq 0$. This means that $\varphi(\phi)$ is a nontrivial periodic function on C_1 , see (4.20).*

3.1 Inverse conformal mapping. According to our contraction (see 1.2) the inverse function $z(w)$ maps C_{ρ_B} into the contour ∂B, i.e. formally $\partial B = \{z(w = \rho_B e^{i\varphi})\}_{\varphi\in[0,2\pi)}$.

Notice that using (4.15) we can introduce the holomorphic function:

$$G(w) := F(z(w))^{-1} = \ln(1/\rho_B)\ \frac{z^{'}(w)}{z(w)}\ w = G_0 + \sum_{n=1}^{\infty}(G_n w^n + G_{-n} w^{-n})\ , \quad (4.23)$$

where the last sum is the corresponding Laurent series. Hence, following the same line of reasoning as in Section 2, we obtain:

$$z(w) = z_0\ w^{G_0/\ln(1/\rho_B)} \exp\left[(\ln(1/\rho_B))^{-1}\sum_{n=1}^{\infty}(G_n w^n - G_{-n} w^{-n})/n\right]\ . \quad (4.24)$$

Notice that on the circle C_1 the function $z(w = e^{i\varphi})$ is periodic. Then the same is true for G. By the arguments similar to those in Section 2, this function has the Fourier coefficients satisfying the same properties as g_n in (4.16), i.e. by (4.23) one gets:

$$G_n = \overline{G_{-n}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi\ G(e^{i\varphi})\ e^{-in\varphi}\ . \quad (4.25)$$

3.2 Localisation of ∂B. Since $z : C_1 \to C_1$, then similar to Section 2, the representation (4.24) for this periodic function implies that we can choose $z_0 = 1$ and that $G_0/\ln(1/\rho_B) = 1$, or $G_0 = 1/g_0$. By virtue of (4.16) and (4.23) the other coefficients are given by

$$G_m = \frac{1}{2\pi i}\int_{C_1} dw\ \frac{1}{w^{m+1}}\ \frac{1}{F(z(w))} = \frac{1}{2\pi}\int_0^{2\pi} d\phi\ \frac{e^{i\phi}}{g(\phi)}\frac{w'(e^{i\phi})}{w^{m+1}(e^{i\phi})}\ . \quad (4.26)$$

Since the conformal mapping w has been already calculated in (4.22) for given Neumann data g, formulae (4.26) solve the problem of inversion $z(w)$, see (4.24).

Hence in the case $f_+ = 1$ and $f_- = 0$ the position of unknown boundary ∂B is defined for a given Neumann data g as a set

$$\partial B = \{z(w = \rho_B e^{i\varphi})\}_{\varphi\in[0,2\pi)}\ , \quad (4.27)$$

which is uniquely defined by (4.24),(4.26) and auxiliary radius $\rho_B = e^{-1/g_0}$.

3.3 Existence and uniqueness. Notice that existence and uniqueness of the solution (4.27) follow from the explicit contraction 3.2. This statement is not connectionless. The first necessary condition has been already mentioned:

(i) $g_0 > 0$, see (4.21).

Another restriction follows directly from the $f_\pm$-boundary conditions for the Problem $(P_{f_\pm = 1,0})$:

(ii) $g(\phi) > 0$, see (4.5) and (4.7).
(iii) A more subtle constrain for the given Neumann data $g(\phi)$ follows from the conditions insuring the invertibility of the conformal mapping w. We study this restriction first for the particular example in the next subsection 4.1.

4.1 Example of the Neumann data. Let

$$g(\phi) = g_0 + 2g_1 \cos\phi \ , \tag{4.28}$$

for $g_0 > 0$ and $g_1 > 0$. Then by (4.22) one immediately gets

$$w(z) = z \ \exp\left[(g_1/g_0)(z - z^{-1})\right] \ , \tag{4.29}$$

but our aim is to inverse the function $w(z)$, i.e. to find (4.24) and then to calculate the unknown boundary ∂B (4.27).

Remark that in spite of $|w(z = e^{i\phi})| = 1$, the conformal mapping (4.29) acts nontrivially on C_1 since:

$$w(e^{i\phi}) = e^{i\phi} \exp\left[2i(g_1/g_0)\sin\phi\right] \ . \tag{4.30}$$

The equation (4.30) yields for the function $\varphi(\phi)$, see (4.20), the expression:

$$\varphi(\phi) = \phi + 2(g_1/g_0)\sin\phi \ . \tag{4.31}$$

4.2 Conditions on the external current. Notice first that general conditions on $g(\phi)$ imply: $g_0 > 0$ and $g_0 > 2g_1$, see (i) and (ii). For example, the importance of $g_0 > 2g_1$ is directly related to *monotonicity* of the function (4.31).

A more delicate condition (iii) requires that $w : \partial B \to C_{\rho_B}$ and in particular:

$$w(z = r(\phi))\,|_{\phi=0} = r(\phi) \ \exp\left[(g_1/g_0)(r(\phi) - r(\phi)^{-1})\right]|_{\phi=0} = \rho_B \tag{4.32}$$

$$w(z = -r(\phi))\,|_{\phi=\pi} = -r(\phi) \ \exp\left[(g_1/g_0)(-r(\phi) + r(\phi)^{-1})\right]|_{\phi=\pi} = -\rho_B \tag{4.33}$$

Notice that for given $g_0 > 0$ and $g_0 > 2g_1$, the solution of (4.32) for $r(\phi = 0)$ always exists and it is unique. Whereas for $r(\phi = \pi)$ it is not true. Indeed, for any $r < 1$ the function defined by the left-hand side of (4.33):

$$F_\varepsilon(r) := r \ \exp\left[\varepsilon(-r + r^{-1})\right] > 0 \quad , \quad \varepsilon := g_1/g_0 < 1/2 \ ,$$

is monotonously increasing, for increasing ε. Hence, there is a critical value $\varepsilon_{cr} : 0 < \varepsilon_{cr} < 1/2$, corresponding to condition

$$\min_{r\leq 1} F_{\varepsilon_{cr}}(r) = \rho_B \ , \tag{4.34}$$

and there are no solutions $r(\phi = \pi) < 1$ of (4.33) for $\varepsilon > \varepsilon_{cr}$. Let $g_0 = 1$. Then one obtains from (4.34) the equation for ε_{cr} in the form:

$$\ln[(1 - \sqrt{1 - 4\varepsilon^2})/2\varepsilon] + \sqrt{1 - 4\varepsilon^2} + 1 = 0 \ . \tag{4.35}$$

Equation (4.35) implies that solution for $r(\phi = \pi)$ does not exist, when $1/2 > g_1$, but $g_1 > g_{cr} = 0,13796148...$. This means that for $g_1 > g_{cr}$ the conformal map w is not invertible, i.e. the image ∂B is not correctly defined.

We illustrate this evolution of conformal mapping and the form of the internal absorbing boundary ∂B as a function of g_1 for $g_0 = 1$ by Figures 1-5.

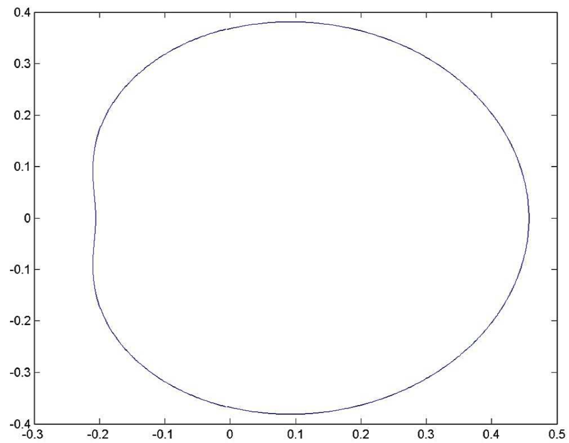

FIGURE 1. Internal boundary ∂B for $g_0 = 1$ and $g_1 = 0,125 < g_{cr}$

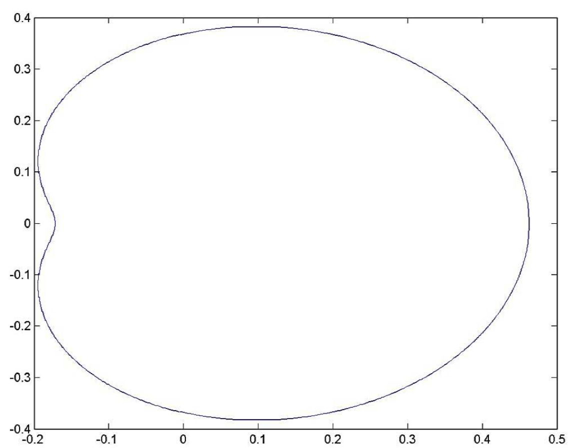

FIGURE 2. Internal boundary ∂B for $g_0 = 1$ and $g_1 = 0,135 < g_{cr}$

On the last two figure one observes that the boundary ∂B is not closed because of small gaps for $\varphi(\phi = \pi) = \pi$, see (4.31). This is a numerical indication that the conformal map w is not invertible for $g_1 > g_{cr}$.

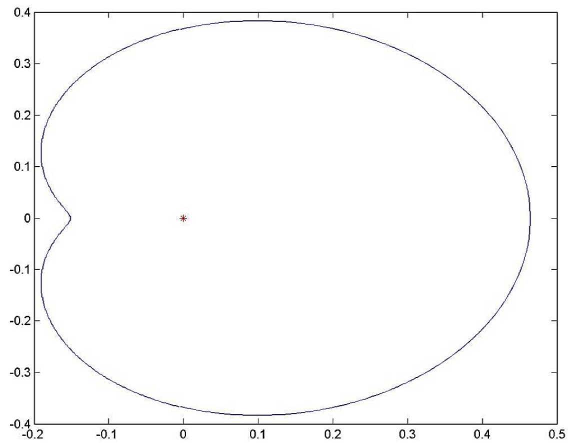

FIGURE 3. Internal boundary ∂B for $g_0 = 1$ and $g_1 = 0,13796148 < g_{cr}$

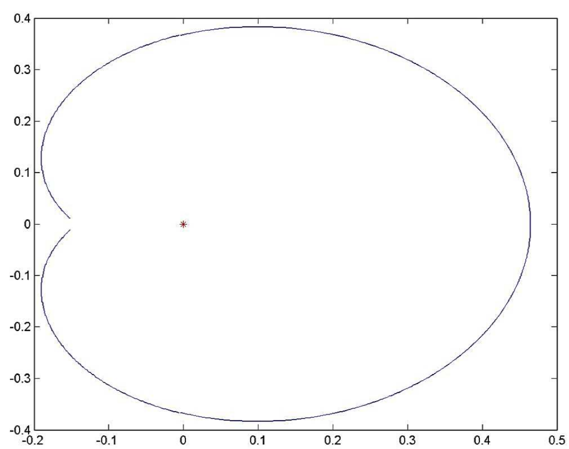

FIGURE 4. Internal boundary ∂B for $g_0 = 1$ and $g_1 = 0,1382 > g_{cr}$

5. Concluding remarks

1. First we comment the case $\alpha = 0$, i.e. the Neumann boundary conditions on the absorbing cell ∂B, see **(P2)**. Then ($\mathbf{P}^{\infty}_{\mathbf{d=2}}$) transforms into the problem

$$(\mathbf{P}^{\alpha=\mathbf{0}}_{\mathbf{d=2}}) \qquad \begin{cases} \Delta u = 0,\ p \in \Omega \setminus \overline{B}\ , \\ u\mid_{\partial\Omega}(p) = f(p), \quad p \in \partial\Omega\ , \\ \partial_{\nu} u \mid_{\partial B}(\omega) = g(\omega), \quad \omega \in \partial B\ . \end{cases}$$

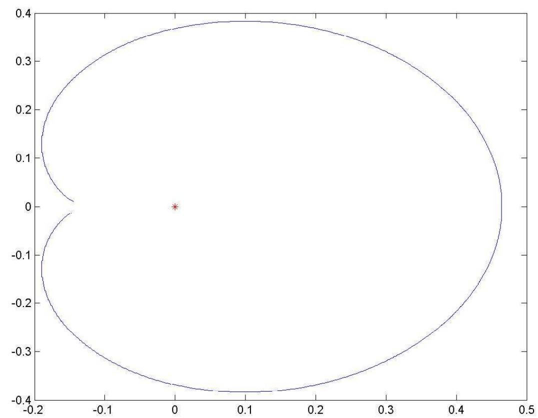

FIGURE 5. Internal boundary ∂B for $g_0 = 1$ and $g_1 = 0,1387 > g_{cr}$

To map domain $\Omega \setminus \overline{B}$ onto annulus (4.3) we use the same holomorphic function $\zeta(z)$. Since conformal mappings preserve angles, the corresponding problem gets the form:

$$(\widetilde{\mathbf{P}}^{\alpha=\mathbf{0}}_{\mathbf{d=2}}) \qquad \begin{cases} \Delta\widetilde{u} = 0, \ p \in A_B \ , \\ \widetilde{u}\mid_{C_1}(p) = \widetilde{f}(p) \ , \quad p \in C_1 \ , \\ \partial_\nu \widetilde{u}\mid_{C_{\rho_B}}(\omega) = |\partial_z\zeta(\omega)|\widetilde{g}(\omega), \quad \omega \in C_{\rho_B} \ . \end{cases}$$

Here $\partial_\nu(\cdot)\mid_{C_{\rho_B}}$ is external *normal* derivative at the point $\omega \in C_{\rho_B} = \zeta(\partial B)$ for a value proportional to $\widetilde{g}(\omega) = (g \circ \zeta)(\omega)$.

It is clear that now our scheme must be considerably modified (simplified), since actual boundary conditions depend on *unknown* conformal mapping ζ. Note that we can not get a help from Proposition 3.3 to reduce the Neumann boundary condition to Dirichlet, since our domain is not simply connected. The external data for solution of the inverse geometrical problem correspond to $\widetilde{f}(p)$. So, we prefer to simplify the conditions on the cell surface ∂B and to put $g = 0$, which excludes the appearance of annoying dependence on derivative $\partial_z\zeta$ of the Neumann boundary condition.

2. Consider the problem $(\widetilde{\mathbf{P}}^{\alpha=\mathbf{0}}_{\mathbf{d=2}})$ for $\widetilde{g} = 0$.

$$(\widetilde{\mathbf{P}}^{\mathbf{0}}_{\mathbf{d=2}}) \qquad \begin{cases} \Delta\widetilde{u} = 0, \ p \in A_B \ , \\ \widetilde{u}\mid_{C_1}(p) = \widetilde{f}(p) \ , \quad p \in C_1 \ , \\ \partial_\nu \widetilde{u}\mid_{C_{\rho_B}}(\omega) = 0 \ , \quad \omega \in C_{\rho_B} \ . \end{cases}$$

Example 5.1. *As above (see Example 3.4) we first illustrate a possible strategy to solve* $(\widetilde{\mathbf{P}}^{\mathbf{0}}_{\mathbf{d=2}})$ *by a simple example of the round Neumann absorbing cell.*

Let boundaries $\partial\Omega = C_R$ and $\partial B = C_{r_B}$ be two concentric circles with radius r_B, which is the only unknown parameter that should be defined as a solution of the inverse geometrical problem. Moreover, since $\zeta : C_{\rho_B} \to \partial B = C_{r_B}$ and $\zeta : C_{r=1} \to \partial\Omega = C_R$, we find this conformal mapping concises with the same linear mapping, $\zeta(z) = R\,z$, as in Example 3.4, i.e. $\rho_B = r_B/R$.

Notice that the constant external condition $\widetilde{f}(p) = (f \circ \zeta)(p) = f(Re^{i\phi}) = f$, $p \in C_1$, implies a trivial constant solution $u_f = \widetilde{u}_f = f$. Therefore, we consider the one-mode boundary condition defined by $\widetilde{f}(e^{i\phi}) = (f \circ \zeta)(e^{i\phi}) = f(Re^{i\phi}) = f(\phi) := \mathrm{f}\cos\phi$. Then by general solution (3.13) in annulus one gets for the Dirichlet-to-Neumann operator, $(\mathbf{P^{\alpha=0}_{d=2}})$ with $g = 0$:

$$\Lambda_{\partial B,\partial\Omega} : f(\phi) \mapsto \partial_\nu u_f \,|_{C_R} = \frac{R^2 - r_B^2}{R\,(R^2 + r_B^2)} f(\phi) \ . \tag{5.1}$$

Similarly one obtains for for the problem $(\widetilde{\mathbf{P}}^{\mathbf{0}}_{\mathbf{d=2}})$:

$$\Lambda_{C_{\rho_B},C_{\rho=1}} : f(\phi) \mapsto \partial_\nu \widetilde{u}_{\widetilde{f}} \,|_{C_1} = \ \frac{1-\rho_B^2}{(1+\rho_B^2)} \ f(\phi) \ . \tag{5.2}$$

By virtue of $\rho_B = r_B/R$, (5.1) and (5.2) imply that relations (3.17) and (3.18), where $l = 2\pi\,R$, are valid with solution (3.24): $\tau_0(\phi) := (l/2\pi)\,\phi\ ,\ \phi \in [0, 2\pi)$.

This example shows that following through verbatim along the arguments of Section 3.4 one obtains the same iterative scheme (3.27)-(3.29), but with Dirichlet-to-Neumann operators that are defined by the Neumann problems $(\mathbf{P^{\alpha=0}_{d=2}})$ and $(\widetilde{\mathbf{P}}^{\mathbf{0}}_{\mathbf{d=2}})$. Example 5.1 gives zero-order approximation for the solution.

3. Recall that the aim of present note is to advocate a *formal* solution of some $d = 2$ inverse geometrical problems, see e.g. Remark 3.5. Since the error in calculations of the coefficients $\{\gamma_s\}_{s\in\mathbb{Z}}$, see (3.30), can be exponentially amplified in expression (3.32) for the boundary ∂B, it is clear that the problem is ill-posed, i.e. it demands some further analysis.

We plan to return to numerical implementations of this formal iterative scheme elsewhere. This needs to study cut-offs and regularizations, as well as their possible generalisations to the Robin boundary conditions,

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